A SIMPLE CALCULATION OF GRAVITATIONAL LENSING ON THE ROTATING STARS

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Abstract

We propose a simple calculation to obtain the gravitational lensing on rotating stars. The calculation has been made by using a Lagrangian of Kerr metric and getting $\delta\phi$ for deflection angle. The calculation has been reduced to slow-rotating stars and equatorials case ($\pi/2$) for maximum deflection angle.

Keywords: *gravitational lensing; rotating stars; Lagrangian; Kerr metric; deflection angle.*

INTRODUCTION

Gravitational lensing was predicted by Einstein's general relativity. After observation led by Eddington in the following years, Eddington proved Einstein's prediction on gravitational lensing. Based on the calculation on gravitational lensing using Schwarzchild metric, the light which is passing through a massive object will be deflected [1] as wide as

$$\Delta\phi = \frac{4GM}{c^2 r_b}, \tag{1}$$

where G, M, c, and r_b respectively are gravitational constant, the mass of the star, the velocity of light, and impact parameter . The detail of this equation is animated by Figure 1.

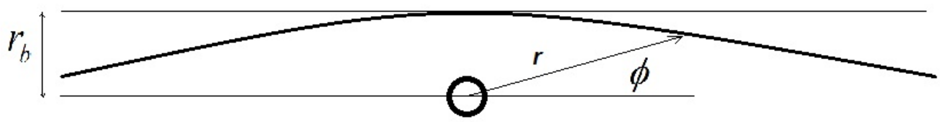

Figure 1. the bending of light by a star.

Recently, there are still can be found many studies on calculating the gravitational lensing especially in the Kerr metric. One of those studies has been done by [2-3] on calculating and comparing the Einstein ring and critical ring of rotating stars. In the following year(s), the magnification in a Kerr gravitational lens has been studied by [4] and [5]. On another side, Bozza and Mancini [6] studied the gravitational lensing on star S2.

In this paper, we calculated the simple analytical case to gravitational lensing on Kerr metric, especially for slow rotating stars in equatorial case. This consideration has also been investigated by [7]. We derive the equation of motion from Lagrangian and use it to calculate the deflection angle by rotating stars.

LAGRANGIAN AND EQUATIONS OF MOTION ON KERR METRIC

In this section, we will derive equations of motion from Lagrangian which is defined by

$$\mathrm{L} \equiv g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} . \tag{2}$$

In order to obtain the full Lagrangian for Kerr metric, we may write Kerr metric to get enough information to fill above Lagrangian as follows

$$\begin{aligned} ds^2 &= c^2\left(1-\frac{2\mu r}{\rho^2}\right)dt^2+\frac{4\mu acr\sin^2\theta}{\rho^2}dtd\phi-\frac{\rho^2}{\Delta}dr^2-\rho^2 d\theta^2 \\ &-\left(r^2+a^2+\frac{2\mu ra^2\sin^2\theta}{\rho^2}\right)\sin^2\theta d\phi^2 . \end{aligned} \tag{3}$$

With $\mu = GM / c^2$, $a = J / Mc$, which J is the angular momentum of a star. In another side, one may notice that $\rho = r^2 + a^2\cos^2\theta$ and $\Delta = r^2 - 2\mu r + a^2$.

By gathering information of from (3), one can construct Lagrangian from Eq. (2)

$$\begin{aligned} \mathrm{L} &= c^2\left(1-\frac{2\mu r}{\rho^2}\right)\dot{t}^2+\frac{4\mu acr\sin^2\theta}{\rho^2}\dot{t}\dot{\phi}-\frac{\rho^2}{\Delta}\dot{r}^2-\rho^2\dot{\theta}^2 \\ &-\left(r^2+a^2+\frac{2\mu ra^2\sin^2\theta}{\rho^2}\right)\sin^2\theta\dot{\phi}^2 . \end{aligned} \tag{4}$$

The dots sign on Eq. (4) shows the differentiation through arbitrary parameter σ and yields the Lagrangian of Kerr metric. This result leads us to obtain Euler-Lagrange equation in the form of

$$\frac{d}{d\sigma}\frac{\partial\mathrm{L}}{\partial\dot{x}^{\mu}}-\frac{\partial\mathrm{L}}{\partial x^{\mu}}=0 . \tag{5}$$

Putting with $\theta = \pi / 2$ which leads $\sin(\pi / 2) = 1$ and $\rho = r$, one obtains

$$\left(1-\frac{2\mu}{r}\right)\dot{t}+\frac{2\mu a}{cr}\dot{\phi}=k , \tag{6}$$

$$\frac{r^2}{\Delta}\ddot{r}-\frac{\mu c^2}{r^2}\dot{t}^2+\frac{2\mu ac}{r^2}\dot{t}\dot{\phi}-\left(r+\frac{\mu a^2}{r^2}\right)\dot{\phi}^2=0 , \tag{7}$$

and $$\frac{2\mu ac}{r}\dot{t}-\left(r^2+a^2+\frac{2\mu a^2}{r}\right)\dot{\phi}=h . \tag{8}$$

One may realise that k and h are constants. We have been reducing our calculation with $\theta = \pi / 2$, which mean we only aim to find the maximum deflection by an object

In our next discussion, we may argue for photon trajectory and obtain $ds^2 = g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = 0$. It means our Lagrangian is also zero in its value. Then we may rewrite Eq. (4) as

$$0 = c^2\left(1-\frac{2\mu}{r}\right)\dot{t}^2 + \frac{4\mu ac}{r}\dot{t}\dot{\phi} - \frac{r^2}{\Delta}\dot{r}^2 - r^2\dot{\theta}^2 - \left(r^2+a^2+\frac{2\mu ra^2}{r^2}\right)\dot{\phi}^2 \quad (9)$$

Our next step will be tricky, by defining

$$\frac{dr}{d\sigma} = \frac{dr}{d\phi}\frac{d\phi}{d\sigma} = \dot{\phi}\frac{dr}{d\phi} \quad (10)$$

and subtituting this result to Eq. (9) (then divided by ϕ^2) resulting

$$\left(\frac{dr}{d\phi}\right)^2 \frac{r^2}{\Delta} = c^2\left(1-\frac{2\mu}{r}\right)\frac{\dot{t}^2}{\dot{\phi}^2} - \frac{4\mu ac}{r}\frac{\dot{t}}{\dot{\phi}} - r^2 - a^2 - \frac{2\mu a^2}{r}. \quad (11)$$

In order to solve this equation, we must roll back the time, and investigate Eq. (6) and (8). By simple mathematics, one may find $\dot{t}$ and $\dot{\phi}$ respectively

$$\dot{t} = \frac{-k\left(r^2+a^2+\frac{2\mu a^2}{r}\right) - \frac{2\mu a}{cr}h}{-\left(1-\frac{2\mu}{r}\right)\left(r^2+a^2+\frac{2\mu a^2}{r}\right) - \frac{4\mu^2a^2}{c^2r^2}} \quad (12)$$

and $$\dot{\phi} = \frac{h\left(1-\frac{2\mu}{r}\right) - \frac{2\mu ac}{r}k}{-\left(1-\frac{2\mu}{r}\right)\left(r^2+a^2+\frac{2\mu a^2}{r}\right) - \frac{4\mu^2a^2}{c^2r^2}}. \quad (13)$$

By looking back at the Eq. (6) and (8), one may guess that generally the value of k is overwhelmed by the value of constant h, means $k << h$. Later, we can drop the value of k in the Eq. (12) and (13), and construct

$$\frac{\dot{t}}{\dot{\phi}} = \frac{-2\mu a}{cr\left(1-\frac{2\mu}{r}\right)}. \quad (14)$$

DEFLECTION ANGLE OF THE ROTATING OBJECT

In this section, we will derive the deflection angle of gravitational lensing by rotating object. The first method we will use here is inserting Eq. (14) into (11), the result is

$$\left(\frac{dr}{d\phi}\right)^2 = \frac{12\mu^2a^2\Delta}{r^4\left(1-\frac{2\mu}{r}\right)} - \Delta - \frac{a^2\Delta}{r^2} - \frac{2\mu a^2\Delta}{r^3}. \quad (15)$$

Later, we will subtitute the value of $\Delta = r^2 - 2\mu r + a^2$, and defining $u = 1/r$ on Eq. (15) yields

$$\left(\frac{du}{d\phi}\right)^2 = \frac{-4\mu^2a^2u^6 + 8\mu^3a^2u^7 - 4\mu^2a^2u^8}{1-2\mu u} - u^2 - 2\mu u^3 + 2a^2u^4 + a^4u^6 - 4\mu^2a^2u^6 + 2\mu a^4u^7 . \tag{16}$$

By differentiating with ϕ once again, we may have u^n which n is an integer. Since $u = 1/r$, higher order of n leads $u \to 0$, then we drop all values which $n > 3$. The result is

$$\frac{d^2u}{d\phi} + u = 3\mu u^2 - 4a^2u^3 . \tag{17}$$

Solution of Eq. (17) should be in the form

$$u = \frac{\sin\phi}{r_b} + \delta u , \tag{18}$$

which $r_b = r\sin\phi$. Our next job will be searching the solution of δu. The mechanism is substituting Eq. (18) into (17), hence we get

$$\frac{d^2\delta u}{d\phi^2} + \delta u = \frac{3\mu}{r_b^2}\sin^2\phi - \frac{4a^2}{r_b^3}\sin^3\phi , \tag{19}$$

and the final solution reveals

$$u = \frac{\sin\phi}{r_b} + \frac{3\mu}{r_b^2}\left(\frac{1}{2} + \frac{1}{6}\cos 2\phi\right) - \frac{4a^2}{r_b^3}\left(\frac{1}{32}\sin 3\phi - \frac{3}{8}\phi\cos\phi\right). \tag{20}$$

Again, we may put $\mu = 3GM/c^2$ then regard $\phi \approx 0$ for small deflection, resulting $\sin\phi \approx \phi, \cos\phi \approx \cos 2\phi \approx 1$, and the last; $u \approx 0$. Lastly, we can obtain total deflection by calculating $\delta\phi = \phi - (-\phi) = 2\phi$, and hence

$$\delta\phi = \frac{4GM}{c^2r_b}\left(1 + \frac{9a^2}{8r_b^2}\right)^{-1} \approx \frac{4GM}{c^2r_b} - \frac{36GMa^2}{8c^2r_b^3} . \tag{21}$$

The first term in Eq. (21) is Schwarzchild deflection which is shown by Eq. (1). The second term appears because of rotation variable a. The negative sign in 2nd term implies that rotation will decrease the total deflection angle $\delta\phi$.

CONCLUSSION

From Eq. (21) we might realised that rotation of a star decrease their deflection rate toward photon. The above discussion is proved that, by simple algebra, we could obtain a good estimation through gravitational lensing on rotating stars. Even though the mathematical method we used is simple, since there are many parameters included, we might drop some terms in some case in order to make it work. This calculation is best suited to be pedagogycally taught to graduate students.

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